

RPN69

**RADIATION PROJECT INTERNAL REPORT NO. 9**  
**Dynamical Principles and Mathematical Methods**  
**Available for the Analysis and Optimization of**  
**Electron Trapping in an Axially Symmetric**  
**Magnetic Converter. I - by E. J. Schrep**

**10 June 1968**

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DYNAMICAL PRINCIPLES AND MATHEMATICAL METHODS AVAILABLE FOR THE  
ANALYSIS AND OPTIMIZATION OF ELECTRON TRAPPING IN AN AXIALLY  
SYMMETRIC MAGNETIC CONVERTER. I

by

E. J. Schremp

1. An Approach to the Solution of the General Converter Problem  
via a Certain Simplified Version of It.

In Radiation Project Internal Report No. 4,<sup>1</sup> there is included a preliminary estimate of the range of trapping of electrons that can be expected when a collimated, approximately mono-energetic beam of these particles, originating from a linac, enters the time-dependent magnetic field of a certain proposed type of magnetic converter. In a magnetic converter of this type, the magnetic field is purposely given a shape which possesses rotational symmetry about an axis perpendicular to, and slightly displaced from, the central axis of the linac beam. An additional characteristic of the magnetic field in a converter of this type is that it possesses reflection symmetry with respect to a plane which contains the central axis of the linac beam and is perpendicular to the axis of rotational symmetry of the magnetic field.

In connection with the general problem of converting the kinetic energy residing in a collimated beam of very fast electrons into electromagnetic radiation through the trapping of these electrons in a magnetic field, the first of the above mentioned two symmetry properties of the electron-trapping magnetic field, that of axial symmetry, deserves special attention\* both from the theorist and from

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\*To a lesser extent, the second of these two symmetry properties, that of reflection symmetry, also deserves special attention.

the experimentalist, because of the fact that, for those types of magnetic converter which incorporate such symmetry, and only for such types, an opportunity arises for the theorist to exploit certain general dynamical principles and special mathematical methods which lead to a vast simplification of an otherwise rather intractable problem: namely, that of analyzing, and securing a sufficiently comprehensive understanding of, the orbital motions of the electrons coming from the linac, once they enter the magnetic field of the converter. The principal purpose of the following discussion will therefore be to call attention to those general dynamical principles and special mathematical methods which are particularly suitable for use in solving the specific dynamical problem that one is faced with when one attempts to analyze and to optimize the trapping of electrons in an axially symmetric magnetic converter.

Strictly speaking, these simplifying dynamical principles and mathematical methods are applicable only to a certain well-defined simplified version of this actual dynamical problem, in which one temporarily neglects the effects of (a) the time dependence of the magnetic field, (b) the damping force on the electrons due to the emission of electromagnetic radiation from them, and (c) the mutual forces between electrons associated with space charge. However, to whatever extent may be necessary, all of these complicating effects can be incorporated afterwards with relative ease, as small perturbations to an already very informative and comprehensive body of approximate results, obtained first, at a minimum cost of time and effort, by solving this simplified version of the actual problem.

For the most part, the present report will be confined to the treatment of this simplified version of the problem. In a later report, we hope to show how our present results, when taken in conjunction with certain necessary small perturbations of the above kinds, will lead to various practical suggestions for optimizing the design of a magnetic converter of the general type under consideration here.

## 2. Precise Definition of the Simplified Version of the Problem.

When, in the above mentioned simplified version of our dynamical problem, the equations of motion of an individual electron in the presence of the magnetic field of the converter are expressed in Lagrangian form, these equations find their most suitable expression in terms of the following set of cylindrical coordinates: the normal distance  $r$  of the electron from the polar axis (i.e., the axis of rotational symmetry) of the magnetic field; the normal distance  $z$  of the electron from the equatorial plane (i.e., the plane with respect to which there is reflection symmetry) of the magnetic field; and the longitude angle  $\varphi$  of the electron's meridian plane (i.e., the plane containing both the electron and the polar axis). The corresponding set of right-handed cartesian coordinates are then the quantities

$$x = r \cos \varphi, \quad (1a)$$

$$y = r \sin \varphi, \quad (1b)$$

and  $z$ ; and the origin  $O$  of these cartesian coordinates  $x, y, z$  is the center of symmetry of the magnetic field. The polar axis of the magnetic field is evidently the  $z$ -axis; and distances  $z$  are reckoned as positive for points above, and negative for points below, the equatorial plane. The longitude angle  $\varphi$ , which in general could be measured relatively to any arbitrary fixed meridian plane, will here be defined, for convenience, by taking the positive  $y$ -axis to have the same direction and sense as does the linac beam. The  $x$ -axis then intersects normally the central axis of the linac beam.

For the simplified version of our dynamical problem that has been specified above, the Lagrangian equations of motion of an individual electron are then of the form

$$\left(\frac{d}{dt}\right)(\partial L / \partial \dot{q}_1) - (\partial L / \partial q_1) = 0, \quad (i = 1, 2, 3) \quad (2)$$

where  $q_1=r$ ,  $q_2=z$ ,  $q_3=\varphi$ , and the dot indicates total differentiation with respect to the time  $t$ . Here, the Lagrangian  $L$  is the appropriate one for a system consisting of an electron of rest mass  $m_0$  and (negative) charge  $e$ , with instantaneous velocity  $\vec{v}$ , subject to no forces other than those arising from a time-independent magnetic field possessing the two symmetry properties previously mentioned.

In the case of a general external electromagnetic field, with vector potential  $\vec{A}$  and scalar potential  $\Phi$ , the appropriate form of the Lagrangian  $L$  would be

$$L = -m_0 c^2 (1 - \beta^2)^{\frac{1}{2}} + (e/c)(\vec{A} \cdot \vec{v}) - e\Phi, \quad (3)$$

where  $c$  is the speed of light,  $\beta=v/c$ , and  $v=|\vec{v}|$ . Since the electric field is zero in the present simplified version of our dynamical problem, we must here set  $\Phi=0$ ; and thus we need compute expressions only for the two quantities  $\vec{A}$  and  $\vec{v}$  in Eq. (3). Owing to the axial symmetry of the magnetic field in our particular problem,\* the vector potential  $\vec{A}$ , when resolved along the three orthogonal unit vectors  $e_r$ ,  $e_z$ ,  $e_\varphi$  at any point, has components  $(0, 0, A_\varphi)$ ; while the particle velocity  $\vec{v}$ , when similarly resolved, has components  $(\dot{r}, \dot{z}, r\dot{\varphi})$ . Accordingly, the appropriate Lagrangian for the present simplified version of our dynamical problem is

$$L = -m_0 c^2 (1 - \beta^2)^{\frac{1}{2}} + (e/c)(r\dot{\varphi}A_\varphi). \quad (4)$$

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\* Since the existence of a nonvanishing  $\varphi$ -component of the electron-trapping magnetic field would constitute an unnecessary complication, we are assuming here that this component is everywhere zero.

### 3. Angular Momentum and Energy Integrals for the Simplified Version of the Problem.

An additional consequence which follows from our assumption of axial symmetry of the magnetic field is that  $A_\varphi$  in Eq. (4) is independent of the longitude angle  $\varphi$ . It is thus seen from Eq. (4) that the coordinate  $\varphi$  does not appear explicitly in the Lagrangian  $L$  for the present simplified version of our dynamical problem. Therefore we obtain immediately an integral of the motion from the equation of motion corresponding to the coordinate  $\varphi$ :

$$(d/dt)(\partial L/\partial \dot{\varphi}) = (\partial L/\partial \varphi) = 0, \quad (5)$$

or

$$(d/dt)p_\varphi = 0, \quad (6)$$

where

$$p_\varphi = mr^2\dot{\varphi} + (e/c)rA_\varphi \quad (7)$$

and where

$$m = m_0/(1-\beta^2)^{\frac{1}{2}} \quad (8)$$

is the relativistic mass of the particle. This first integral of the motion expresses the fact that there is conservation of the  $\varphi$ -component of generalized angular momentum  $p_\varphi$ , conjugate to the ignorable coordinate  $\varphi$ .

Upon definition of the Hamiltonian  $H$  for the present simplified version of our problem as

$$H = -L + \sum_i \dot{q}_i (\partial L/\partial \dot{q}_i) \quad (9)$$

and upon use of the equations of motion and the assumption that the time  $t$  does not enter explicitly into the Lagrangian  $L$ , it follows that there is conservation of energy:

$$dH/dt = -\partial L/\partial t = 0. \quad (10)$$

Accordingly, an energy integral exists, as follows:

$$(d/dt)H = 0, \quad (11)$$

where

$$H = mc^2. \quad (12)$$

It is clear that  $v$  and  $m$  are thereby constant for the present simplified version of our problem.

4. Reduction of the Simplified Version of the Problem to a Dynamical Problem of Two Degrees of Freedom.

Following the procedure introduced in Radiation Project Progress Report No. 3,<sup>2</sup> let us now define a function  $\Psi(r,z)$  such that

$$\Psi = rA_{\varphi}; \quad (13)$$

and let us also define a constant  $\Psi_0$  such that

$$p_{\varphi} = (e/c)\Psi_0. \quad (14)$$

Then Eq. (7) may be re-expressed in the form

$$mr^2\dot{\phi} = (e/c)(\Psi_0 - \Psi). \quad (15)$$

On dividing Eq. (15) through by the constant quantity  $mv$ , one obtains the equation

$$r^2(d\phi/ds) = (\Psi_0 - \Psi)/\rho, \quad (16)$$

where

$$\rho = mvc/e \quad (17)$$

is the so-called rigidity of the particle, and where

$$(ds/dt)^2 = v^2 = \dot{r}^2 + \dot{z}^2 + r^2\dot{\phi}^2, \quad (18)$$

the new independent variable  $s$  being the arc length measured along an orbit. It is seen that Eq. (16) describes the motion of the meridian plane containing the particle, for specified values of the rigidity  $\rho$  of the particle and of its angular momentum at infinity,  $p_\phi = (e/c)\Psi_0$ , about the polar axis of the magnetic field.

From Eq. (18) it follows that

$$(dr/ds)^2 + (dz/ds)^2 + r^2(d\phi/ds)^2 = 1. \quad (19)$$

If we now write

$$Q = (dr/ds)^2 + (dz/ds)^2, \quad (20)$$



then we find from Eq. (19) that

$$Q = 1 - r^2(d\phi/ds)^2; \quad (21)$$

and hence we have from Eq. (16) the result that

$$Q = 1 - (\Psi_0 - \Psi)^2 / \rho^2 r^2. \quad (22)$$

On comparing Eqs. (20) and (22), we now observe that there are two equivalent ways in which it is possible to express the derivative of  $Q$  with respect to  $s$ , namely:

$$dQ/ds = 2(dr/ds)(d^2r/ds^2) + 2(dz/ds)(d^2z/ds^2) \quad (23)$$

and

$$dQ/ds = (\partial Q/\partial r)(dr/ds) + (\partial Q/\partial z)(dz/ds), \quad (24)$$

the latter way being made possible through the explicit functional dependence of  $Q$  upon  $r$  and  $z$ , as specified in Eq. (22). It follows, therefore, that the following pair of equations must hold:

$$d^2r/ds^2 = \frac{1}{2}(\partial Q/\partial r), \quad (25a)$$

$$d^2z/ds^2 = \frac{1}{2}(\partial Q/\partial z). \quad (25b)$$

Now the position of the particle in its own meridian plane is defined by the rectangular coordinates  $r$  and  $z$ ; and therefore we

see, from Eqs. (25), that the motion of the particle in its own meridian plane is exactly the same as if the particle were moving under the influence of a potential  $-Q/2$ , where  $Q$  is given by Eq. (22) for specified values of the rigidity  $\rho$  of the particle and of its angular momentum at infinity,  $p_\phi = (e/c)\Psi_0$ , about the polar axis of the magnetic field.

Thus, for any specified values of the rigidity  $\rho$  and of the angular momentum at infinity,  $p_\phi = (e/c)\Psi_0$ , the problem of determining the corresponding particle orbits has been shown to have been resolved into the following two problems: first, that of determining the motion of the meridian plane containing the particle; and second, that of determining the motion of the particle in its own meridian plane. The solution of the first problem merely requires the integration of Eq. (16); while the solution of the second problem requires the integration of Eqs. (25), with the potential  $-Q/2$  defined by Eq. (22). It is this second problem -- a dynamical problem of two degrees of freedom -- which will be of primary interest to us in what follows.

##### 5. Allowed and Forbidden Regions of a Particle for the Simplified Version of the Problem.

We now have reached the point where it will become quite evident how considerable the advantages are which accrue from the initial requirement that the electron-trapping magnetic field possess certain definite symmetry properties, and from the further decision to neglect, temporarily, the three small complicating effects of space charge, of radiation by the electrons, and of the time-dependence of the magnetic field. Principal among these advantages is the fact that, for the resulting simplified version of our dynamical problem, the exact equations of motion can be used to draw some general conclusions about the confinement of electrons in the magnetic field of the converter. For a magnetic field

possessing the previously assumed degree of symmetry, it is possible very easily to determine certain forbidden regions of a particle, even without knowing the detailed solution of its orbit.

Thus, for whatever values of  $r$  and  $z$  the function  $Q(r,z)$  defined by Eq. (22) becomes negative, we know from Eq. (20) that  $dr/ds$ , or  $dz/ds$ , or both, will take on imaginary values. In other words, the locus of points in the meridian plane defined by the equation

$$Q(r,z) = 0 \quad (26)$$

separates the meridian plane into two types of region: allowed regions for the motion of a particle, for which  $Q(r,z) \geq 0$ ; and forbidden regions for the motion of a particle, for which  $Q(r,z) < 0$ . From the explicit form of Eq. (22), we see that the boundary curves in the meridian plane which separate the allowed regions from the forbidden regions satisfy one or the other of the two equations

$$\Psi = \Psi_0 \pm \rho r. \quad (27)$$

Now it will be recalled from Eq. (13) that the function  $\Psi(r,z)$  serves to specify completely the magnetic field of the converter; and it is clear that this function can be assumed to be completely known<sup>2</sup> for a magnetic converter of any given specific design and of the general type under consideration here. Accordingly, the determination of the boundary curves defined by Eqs. (27) is seen to be a very straightforward and simple matter: one has only to consider the single general surface  $\Psi(r,z)$ , erected in the space with coordinates  $r,z,\Psi$ , in conjunction with the two planes

$$\Psi_{\pm}(r) = \Psi_0 \pm \rho r, \quad (28)$$

erected in the same space; and one then has only to look for the locus

of points with coordinates  $(r,z)$  in the  $r$ - $z$  plane which correspond (under orthogonal projection upon that plane) to points of intersection of the general surface  $\Psi(r,z)$  with one or the other of these two planes  $\Psi_{\pm}(r)$ .

Not only is it an extremely simple matter to determine these boundary curves for a particle of given rigidity  $\rho$  and of given angular momentum  $p_{\varphi} = (e/c)\Psi_0$ ; but also it is particularly easy to follow the changes that occur in the forms of these boundary curves, as the rigidity  $\rho$  and the angular momentum  $p_{\varphi} = (e/c)\Psi_0$  of the particle are varied: one then has only to change the (equal and opposite) slopes  $\pm \rho$  and the (equal) intercepts  $\Psi_0$  of the two planes  $\Psi_{\pm}(r)$ , in a systematic manner, and one can then obtain a completely comprehensive understanding of the allowed and forbidden regions of a particle, simply from a knowledge of the detailed shape of the known magnetic field function  $\Psi(r,z)$ .

6. Circular Orbits in the Simplified Version of the Problem, and Orbits Asymptotic to Them.

We turn now to a consideration of the important question concerning the existence of circular orbits in the simplified version of our dynamical problem, and of orbits asymptotic to them. As we shall presently see, a knowledge of the existence and nature of these orbits will be indispensable to us, first, in systematizing our general understanding of this dynamical problem, and, second, in enabling us to exploit these systematized results by applying them to the problem of optimizing the design of a magnetic converter.

It is easy to see that, in general, a circular orbit, if it exists for a given magnetic field possessing the previously mentioned symmetry properties, must satisfy the three equations

$$r = r_0, \quad (29a)$$

$$z = z_0, \quad (29b)$$

$$r_0 (d\phi/ds) = \pm 1, \quad (29c)$$

where  $r_0$  and  $z_0$  are constants. Now the first two of these equations evidently imply also the equations

$$dr/ds = dz/ds = d^2r/ds^2 = d^2z/ds^2 = 0; \quad (30)$$

and hence, as may be seen from Eqs. (20) and (25), they further imply the equations

$$Q = \partial Q/\partial r = \partial Q/\partial z = 0. \quad (31)$$

According to Eq. (26), therefore, for every circular orbit that is found to exist, the corresponding point with coordinates  $(r_0, z_0)$  in any meridian plane must lie on one of the boundary curves in that meridian plane which separate the allowed regions for the motion of a particle from the forbidden regions.

But, more than this, we also see from the last two of Eqs. (31) that every circular orbit is further characterized by the fact that the corresponding point with coordinates  $(r_0, z_0)$  in the meridian plane is a critical point<sup>3</sup> of the function  $Q(r, z)$ ; i.e., either it is a point at which the function  $Q(r, z)$  attains a relative maximum or minimum value, or else it is a saddle point of the function  $Q(r, z)$ . Now, at any such critical point, as has just been expressed in the first of Eqs. (31), the value of the function  $Q(r, z)$  must be zero. Therefore, if at this critical point the function  $Q(r, z)$  attains a relative maximum or minimum value, the corresponding boundary curve between allowed and forbidden regions on which this critical point lies is then just a degenerate locus consisting of only one point. If, on the other hand, this critical point is a saddle point of the function  $Q(r, z)$ , then it must be a point of intersection between

two nondegenerate continuous boundary curves, each of which separates allowed from forbidden regions. This latter situation, in its most general form, is shown schematically in Fig. 1.

Now in all situations of the latter kind, the circular orbit in question is unstable; and a characteristic feature of this kind of unstable circular orbit is the concomitant existence of two distinct families of orbits which are asymptotic to each such circular orbit, one of these two families of asymptotic orbits being confined to one of the two allowed regions shown schematically in Fig. 1, and the other family being confined to the other allowed region. This characteristic feature of unstable circular orbits of this kind constitutes the first example of certain fundamental results which are to be drawn, here and in the sequel, from that relatively unfamiliar discipline in dynamics which is known as the "general theory of orbits".<sup>4</sup> It is, however, a feature which, together with certain generalizations of it, we shall find to be most useful in connection with the problem of optimizing the design of a magnetic converter.

In Radiation Project Internal Report No. 4,<sup>1</sup> this particular kind of unstable circular orbit has already been introduced (on the basis of somewhat different, but essentially equivalent, considerations) as the fundamental element of a proposed practical method for the trapping of electrons whose motion is confined to the equatorial plane. In this case, the circular orbit in question is itself located in the equatorial plane ( $z_0=0$ ), and the general situation shown in Fig. 1 reduces to the special situation shown schematically in Fig. 2.

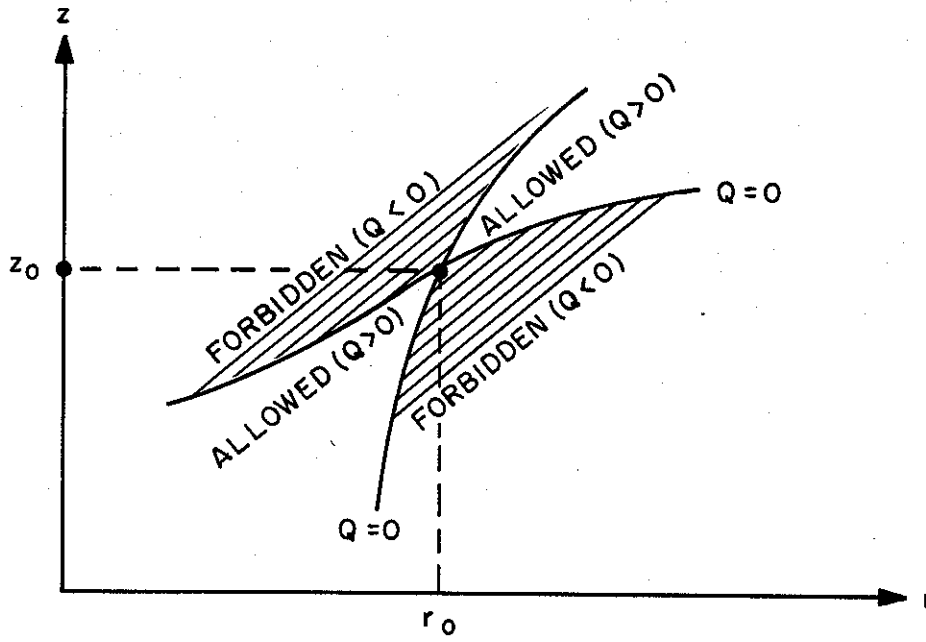


Figure 1

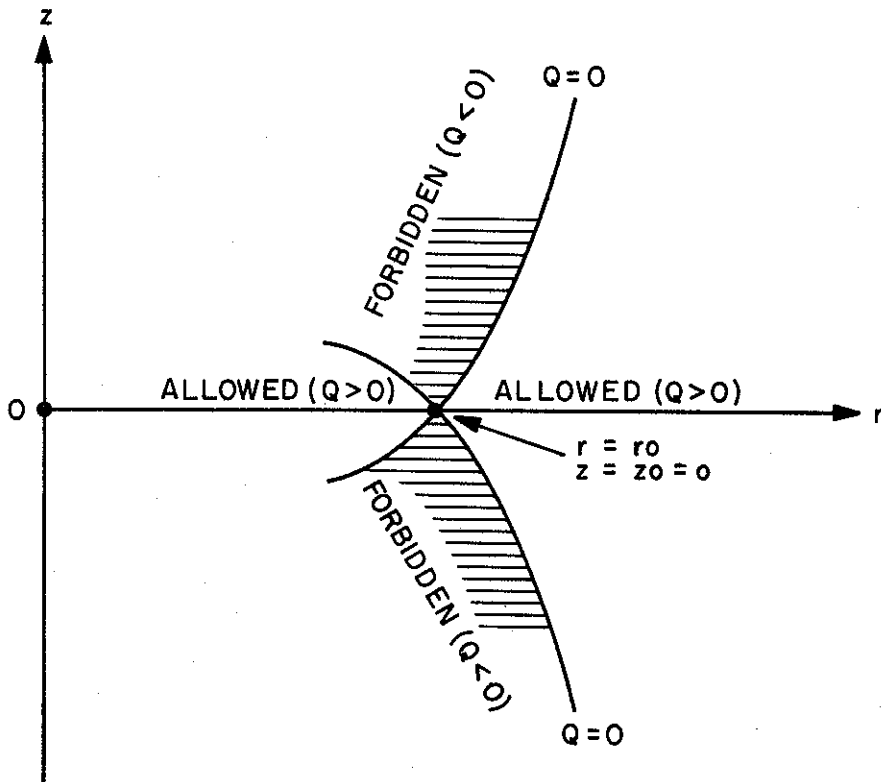


Figure 2

## References

1. D. C. dePackh, "Magnetic Converter Design Considerations," Radiation Project Internal Report No. 4, 6 February 1968.
2. Richard T. Close, "Computer-Assisted Magnetic Field Design, Part I," Radiation Project Progress Report No. 3, 7 March 1968.
3. See Marston Morse, The Calculus of Variations in the Large, American Mathematical Society Colloquium Publications, Volume XVIII, pp. 142-143.
4. See, for example, E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press (Third Edition, 1927), Chapter XV.